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Dynamics of an Axially Moving Beam Submerged in a Fluid

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The dynamics of an axially moving slender beam having a uniform circular cross section and immersed in an incompressible fluid is analyzed. The equation for small transverse vibrations is obtained, taking into account the inviscid and viscous fluid dynamic forces. The transverse displacement is expanded in terms of a set of time-dependent admissible functions to yield a set of nonautonomous ordinary differential equations. Uniform and exponential extension of the beam are considered, and the truncated set of equations is integrated numerically. The system parameters are varied for the two procedures and their influence on stability is noted.

Nomenclature

A	= area of cross section of the beam
$a_{ns}, b_{ns}, c_{ns}, d_{ns}, e_{ns}$	= constants, Eq. (11)
C_D, C_f, C_m	= coefficients associated with form and friction drag and added inertia, respectively
$C_N, C_T, \tilde{C}_N, \tilde{C}_T, \bar{C}_N$	= coefficients derived from C_D and C_f
D	= diameter of the beam
E	= modulus of elasticity $= E_0\left(1+\gamma\frac{\partial}{\partial t}\right)$
F_N, F_L	= viscous forces along normal and longitudinal directions, respectively
I	= moment of inertia of the cross section
i	= angle of incidence
l_0	= original length of the beam
l	= instantaneous length of the beam
M, m	= masses of fluid and beam per unit length, respectively
Q	= shear force
T	= axial tension
u	= transverse displacement of a beam element
$\beta, \bar{\gamma}, \Gamma, \psi, \epsilon$	= dimensionless parameters, Eq. (12)
γ	= coefficient of material damping of the beam
τ	= dimensionless time
λ_n	= eigenvalues of a cantilever beam
ϕ_n	= admissible functions—chosen identical to the eigenfunctions of a cantilever beam

Introduction

THE dynamics of axially moving strings and beams has recently received considerable attention. Earlier works were motivated by the vibrations of moving threadlines in the textile industry and the dynamics of band saws, belts, and chains in mechanical machinery. A review of the literature on this and related areas is given by Mote.[1] A majority of recent investigations on axially moving beams are connected with the development of flexible appendages deployed from spacecraft. Barakat[2] has investigated the linearized transverse vibrations of a moving thin rod between two supports while Tabarrok et al.[3] have analyzed the dynamics of an axially moving cantilever, predicting the possibility of large motion during the later phase of extension. Subsequently, Tabarrok and El Maraghy[4] have studied the stability of an axially oscillating beam for various combinations of the excitation frequency and amplitude. Jankovic[5] investigated the contribution of the second mode to the deflection of a flexible rod during extension and found an approximate solution using the first two modes. Lips and Modi[6] considered the deployment of a flexible appendage from an orbiting spacecraft and noticed that large oscillations resulted for certain deployment rates. On the other hand, the dynamics of fixed-length tubular structures subjected to internal and/or external axial flow has aroused considerable interest due to their applications to oil conveying pipes, heat exchangers, and nuclear fuel rods. It has been shown that for sufficiently large flow velocities a cylinder may be subjected to buckling and flutter instabilities in its first and higher flexural modes, respectively. A review of the existing literature on this topic has been carried out by Paidoussis.[7]

The present study involves both phenomena: fluid-solid interaction and extension of flexible rods. The objective is to examine the dynamics of a flexible slender cantilever beam with uniform circular cross section, extending axially in a horizontal direction at a known rate while immersed in an incompressible fluid. Its stability is investigated for two different types of extension and the effects of the various system parameters are examined. Subsequently the differences in the instability modes between this case and the case of constant-length cylindrical cantilever in axial flow are discussed.

Development of the Equation of Motion

Consider a cylindrical cantilever beam extending axially in a horizontal direction at a rate $\dot{l}$ while immersed in an incompressible fluid of density ρ (Fig. 1). The beam has uniform cross-sectional area A with moment of inertia I and mass per unit length m, while the modulus of elasticity of the material is $E_0(1+\gamma\,\partial/\partial t)$, where γ is a small parameter characterizing the structural damping in the beam. Small lateral motions are considered, and it is assumed that no separation occurs in crossflow and that the fluid forces on each element of the beam are the same as those acting on a corresponding element of a long undeformed beam of the same cross-sectional area and inclination. The fluid is assumed to be contained by boundaries sufficiently distant from the cylinder to have negligible influence on its motion. If $u(x, t)$ is the displacement of an element of the beam, then the resultant relative velocity between the moving beam and the fluid is given by

$$v(x,t)=\frac{\partial u}{\partial t}+\dot{l}\frac{\partial u}{\partial x} \tag{1}$$

In the vicinity of the beam the lateral flow may be considered to be a two-dimensional potential flow with velocity $v(x, t)$. The momentum of the fluid is Mv per unit length of the beam,

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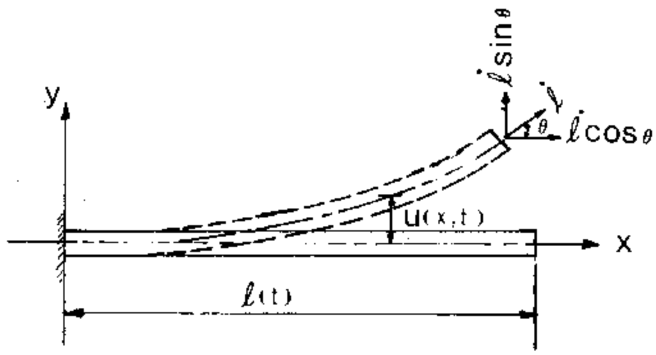

Fig. 1 Geometry of motion.

where M is the virtual mass of the fluid per unit length and is equal to $\rho A C_m$. Here C_m is the added inertia coefficient and is equal to 1 provided that the wavelength of motion is large in comparison with the diameter of the beam.[8] The rate of change of this momentum per unit length is $(\partial/\partial t + \dot{l}\,\partial/\partial x)(Mv)$ and gives rise to an equal and opposite lateral force on the beam.

Consider a small element δx of the beam as shown in Fig. 2. Let the axial tension be denoted by T and the viscous forces per unit length in the normal and longitudinal directions by F_N and F_L, respectively. Then the equation of motion in the two directions may be written as

$$-F_L - m\ddot{l} + \frac{\partial T}{\partial x} = 0 \tag{2a}$$

$$\frac{\partial Q}{\partial x} - F_N - F_L\frac{\partial u}{\partial x} - (M+m)\left(\frac{\partial}{\partial t} + \dot{l}\frac{\partial}{\partial x}\right)^2 u + \frac{\partial}{\partial x}\left(T\frac{\partial u}{\partial x}\right) = 0 \tag{2b}$$

For small lateral motions components of lateral viscous and inertia forces along the axial direction are of second order and have been neglected. Similarly, for horizontal extension, the influence of gravity on the dynamics is of second order and can be ignored. Furthermore, the effects of shear deformation and rotary inertia of the element are neglected so that the shear force Q is given by

$$Q = -EI\frac{\partial^3 u}{\partial x^3} \tag{2c}$$

From Taylor,[9] the viscous forces per unit length of the beam along normal and longitudinal directions are

$$F_N = \tfrac{1}{2}\rho D \dot{l}^2 (C_D \sin^2 i + C_f \sin i) \tag{3a}$$

$$F_L = \tfrac{1}{2}\rho D \dot{l}^2 C_f \cos i \tag{3b}$$

where D is the beam diameter, C_D and C_f are the coefficients associated with form and friction drag for a cylinder in cross-flow, respectively, and i is the angle of incidence given by

$$i = \sin^{-1}(v/\dot{l}) \cong v/\dot{l} \tag{3c}$$

Substituting Eq. (1) into Eq. (3) and linearizing one obtains

$$F_N = \frac{1}{2}C_N\left(\frac{M}{D}\right)\dot{l}\left(\frac{\partial u}{\partial t} + \dot{l}\frac{\partial u}{\partial x}\right) + \frac{1}{2}\tilde{C}_N\frac{M}{D}\left(\frac{\partial u}{\partial t} + \dot{l}\frac{\partial u}{\partial x}\right) \tag{4a}$$

$$F_L = \tfrac{1}{2}C_T(M/D)\dot{l}^2 \tag{4b}$$

where

$$C_N = C_T = (4/\pi)C_f \tag{4c}$$

$$\tilde{C}_N = (4/\pi)C_D(8v_{\max}/3\pi) \tag{4d}$$

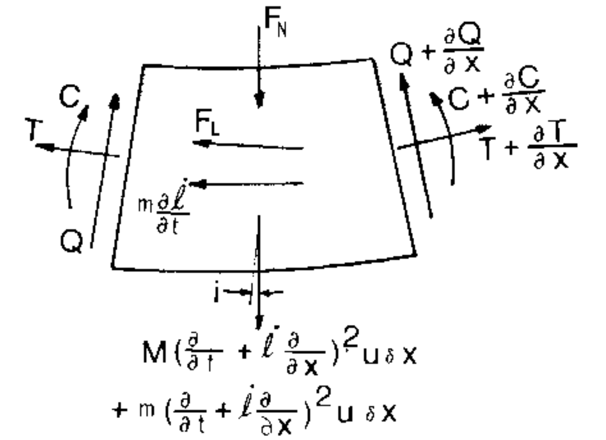

Fig. 2 Equilibrium of forces on a beam element.

The axial tension may be found by substituting F_L into Eq. (2a) and integrating from $x=0$ to $x=l$ to yield

$$T(x) = -\tfrac{1}{2}(M/D)\dot{l}^2 C_T(l-x) - m\ddot{l}(l-x) + T(l) \tag{5a}$$

A nonzero value of $T(l)$ may arise from form drag at the free end, which may be considered proportional to $\tfrac{1}{2}\rho\dot{l}^2 A$, i.e.,

$$T(l) = \tfrac{1}{2}\tilde{C}_T M\dot{l}^2 \tag{5b}$$

Hence

$$T(x) = -\tfrac{1}{2}C_T(M/D)\dot{l}^2(l-x) + \tfrac{1}{2}\tilde{C}_T M\dot{l}^2 - m\ddot{l}(l-x) \tag{6}$$

Substituting Eqs. (2c), (4), and (6) in Eq. (2b) the equation of small lateral motions becomes

$$E_0 I\left(1+\gamma\frac{\partial}{\partial t}\right)\frac{\partial^4 u}{\partial x^4} + \left[(m+M)\dot{l}^2 + \frac{1}{2}C_T\frac{M}{D}\dot{l}^2(l-x) - \frac{1}{2}\tilde{C}_T M\dot{l}^2 + m\ddot{l}(l-x)\right]\frac{\partial^2 u}{\partial x^2} + (m+M)\dot{l}\left(\frac{\partial^2 u}{\partial x\partial t} + \frac{\partial^2 u}{\partial t\partial x}\right)$$
$$+ \left[\frac{1}{2}(C_N\dot{l} + \tilde{C}_N)(M/D)\dot{l} + M\ddot{l}\right]\frac{\partial u}{\partial x} + (m+M)\frac{\partial^2 u}{\partial t^2} + \frac{1}{2}\frac{M}{D}(C_N\dot{l} + \tilde{C}_N)\frac{\partial u}{\partial t} = 0 \tag{7}$$

Analysis

It is convenient to expand the transverse vibrations in series form in terms of a set of admissible functions:

$$u(x,t) = \sum_{n=1}^{\infty}\phi_n(x,l)f_n(t) \tag{8}$$

The functions $\phi_n(x,l)$ can be somewhat arbitrary, as long as they satisfy at least the geometric boundary conditions. In the present analysis,

$$\phi_n(x,l) = [\cosh(\lambda_n x/l) - \cos(\lambda_n x/l)] - \sigma_n[\sinh(\lambda_n x/l) - \sin(\lambda_n x/l)] \tag{9a}$$

where

$$\sigma_n = (\cosh\lambda_n + \cos\lambda_n)/(\sinh\lambda_n + \sin\lambda_n) \tag{9b}$$

and λ_n are the eigenvalues given by the roots of the transcendental equation

$$1 + \cosh\lambda_n\cos\lambda_n = 0 \tag{9c}$$

Substituting Eqs. (8) and (9) in Eq. (7), multiplying the result by $\phi_s(x,l)$, and integrating from 0 to l, one obtains

$$\begin{aligned}\ddot{f}_s = \sum_{n=1}^{\infty} & -[\{[E_0 I\gamma\lambda_n^4/(M+m)l^4] + \tfrac{1}{2}[M/(M+m)D](C_N\dot{l}+\tilde{C}_N)\}\delta_{ns} + 2(\dot{l}/l)(a_{ns}-c_{ns})]\dot{f}_n \\ & + [\{[E_0 I\gamma\dot{l}\lambda_n^4/(M+m)l^5] + (\ddot{l}/l) - (\dot{l}/l)^2 + \tfrac{1}{2}[M/(M+m)D](\dot{l}/l)(C_N\dot{l}+\tilde{C}_N)\}c_{ns} \\ & + \{[M/(M+m)][(\ddot{l}/l) + \tfrac{1}{2}C_T(\dot{l}^2/lD)] + 2(\dot{l}/l)^2\}d_{ns} - (\dot{l}/l)^2 e_{ns} - [E_0 I\lambda_n^4/(M+m)l^4]\delta_{ns} \\ & + \{\tfrac{1}{2}(C_N\dot{l}+\tilde{C}_N)[M/(M+m)D](\dot{l}/l) - (\dot{l}/l)^2 + [M/(M+m)](\ddot{l}/l)\}a_{ns} \\ & - \{\tfrac{1}{2}[C_T(l/D) - \tilde{C}_T][M/(M+m)](\dot{l}/l)^2 + (\dot{l}/l)^2 + [m/(M+m)](\ddot{l}/l)\}b_{ns}]f_n \qquad (s=1,2,\ldots\infty) \end{aligned} \tag{10}$$

where δ_{ns} is the Kronecker delta and

$$a_{ns} = \int_0^l \frac{\partial\phi_n}{\partial x}\phi_s \mathrm{d}x \qquad b_{ns} = l\int_0^l \frac{\partial^2\phi_n}{\partial x^2}\phi_s \mathrm{d}x \qquad c_{ns} = \frac{1}{l}\int_0^l x\frac{\partial\phi_n}{\partial x}\phi_s \mathrm{d}x \qquad d_{ns} = \int_0^l x\frac{\partial^2\phi_n}{\partial x^2}\phi_s \mathrm{d}x \qquad e_{ns} = \frac{1}{l}\int_0^l x^2\frac{\partial^2\phi_n}{\partial x^2}\phi_s \mathrm{d}x \tag{11}$$

It is more convenient to describe the dynamics in terms of dimensionless quantities by defining

$$\tau = [E_0 I/(M+m)l_0^4]^{1/2} t = \beta t \qquad R = l/l_0 \qquad \epsilon = l_0/D \qquad \Gamma = m/M \qquad \psi = 1/(1+\Gamma) \qquad \bar{\gamma} = \gamma\beta \qquad \bar{C}_N = \tilde{C}_N/\beta l_0 \tag{12}$$

where l_0 is the original length of the beam. With these, Eq. (10) becomes

$$\begin{aligned} f_s'' = \sum_{n=1}^{\infty} & -\{[(\bar{\gamma}\lambda_n^4/R^4) + \tfrac{1}{2}\epsilon\psi(C_N R' + \bar{C}_N)]\delta_{ns} + 2(R'/R)(a_{ns}-c_{ns})\}f_n' \\ & + \{[(\bar{\gamma}R'\lambda_n^4/R^5) + (R''/R) - (R'/R)^2 + \tfrac{1}{2}\epsilon\psi(R'/R)(C_N R' + \bar{C}_N)]c_{ns} + [\psi\Gamma(R''/R) + \tfrac{1}{2}C_T\epsilon\psi(R'^2/R) + 2(R'/R)^2]d_{ns} \\ & - (R'/R)^2 e_{ns} - (\lambda_n^4/R^4)\delta_{ns} - [\tfrac{1}{2}(C_N R' + \bar{C}_N)\epsilon\psi(R'/R) + \psi(R''/R) - (R'/R)^2]a_{ns} \\ & - [\tfrac{1}{2}(C_T R\epsilon - \bar{C}_T)\psi(R'/R)^2 + \psi\Gamma(R''/R) + (R'/R)^2]b_{ns}\}f_n \qquad (s=1,2,\ldots\infty) \end{aligned} \tag{13}$$

where prime denotes differentiation with respect to τ. This infinite set of ordinary differential equations can be put in matrix form as follows:

$$\{f''\} + [P(\tau)]\{f'\} + [S(\tau)]\{f\} = \{0\} \tag{14}$$

Since the coefficients $[P(\tau)]$ and $[S(\tau)]$ are time-varying, the usual eigenvalue procedure for studying stability is not applicable. Furthermore, as the coefficients are nonperiodic, Floquet's theory and Bolotin's method are also not useful. Therefore this set of equations, after truncation to four generalized displacements f_n, was integrated numerically for a given $l(t)$. Two types of axial extension are considered: 1) uniform extension, i.e., $\dot{l}$ is a constant, and 2) exponential extension, where $\dot{l}$ is initially large and is reduced exponentially or vice versa.

Results and Discussion

Uniform Extension

Typical behavior of the coefficients f_n in the expansion of the transverse displacement is shown in Fig. 3. The rate of extension is characterized by $R' = 0.1$, while the other parameters are $\bar{\gamma} = 0$, $\psi = 0.5$, $\epsilon C_N = \epsilon C_T = 1$, $\bar{C}_T = 1$ and $\epsilon\bar{C}_N = 0.1$. The oscillations in the first four modes for this particular case remain stable within the range of integration, i.e., until twenty times the original length is reached. It must be emphasized that for nonautonomous systems such as the present case, there are no classical modes; but for the sake of brevity, the admissible functions have been denoted here as "modes." As the length increases at this small extension rate ($R' = 0.1$), the oscillations are damped and the frequencies of oscillation are reduced. Toward the end of integration, the first mode frequency is reduced sufficiently to make the motion overdamped. Figure 4 shows the behavior of f_1 and f_2 for a higher extension rate ($R' = 0.5$). Note that there is no noticeable damping of the motions. When the extension rate is increased to $R' = 1.0$, the beam becomes unstable before twenty times the original length is attained (Fig. 5). The first two modes involve oscillatory motions changing to a static divergence type of instability. However, the third and fourth modes show an oscillatory instability (flutter). Note that the scales in Fig. 3 are different from those in Figs. 4 and 5. A significant amount of numerical experimentation is required to determine exactly the minimum extension rate beyond which the instability sets in, and this is not attempted here. Instead, specific extension rates are considered and the behavior of the beam as well as incipience of instability are noted. It is difficult to determine the length or time when the beam becomes unstable. However, for the sake of comparison, the dimensionless length at which f_n crosses the equilibrium configuration for the last time before becoming unstable has been defined as the length of instability. This definition is useful in the case of divergence type instability associated with the first mode which is the most critical one. A set of uniform dimensionless velocity R' of the beam and the corresponding dimensionless length R when the first mode instability starts to occur are shown in Table 1. As may be expected, the higher the extension rate, the lower the length of instability.

It is interesting to compare these results with those of Paidoussis[10] who considered the dynamics of a fixed-length cylindrical beam in uniform axial flow. For the same parameters as above, the critical dimensionless flow velocity was found to be 2.6, which translates to $R' = 3.67$ based on initial length. In the present case much smaller R' (i.e. $R' = 1$) can lead to instability. For a pressurized neutrally buoyant telescopic appendage made out of polyethylene and having a 3 cm

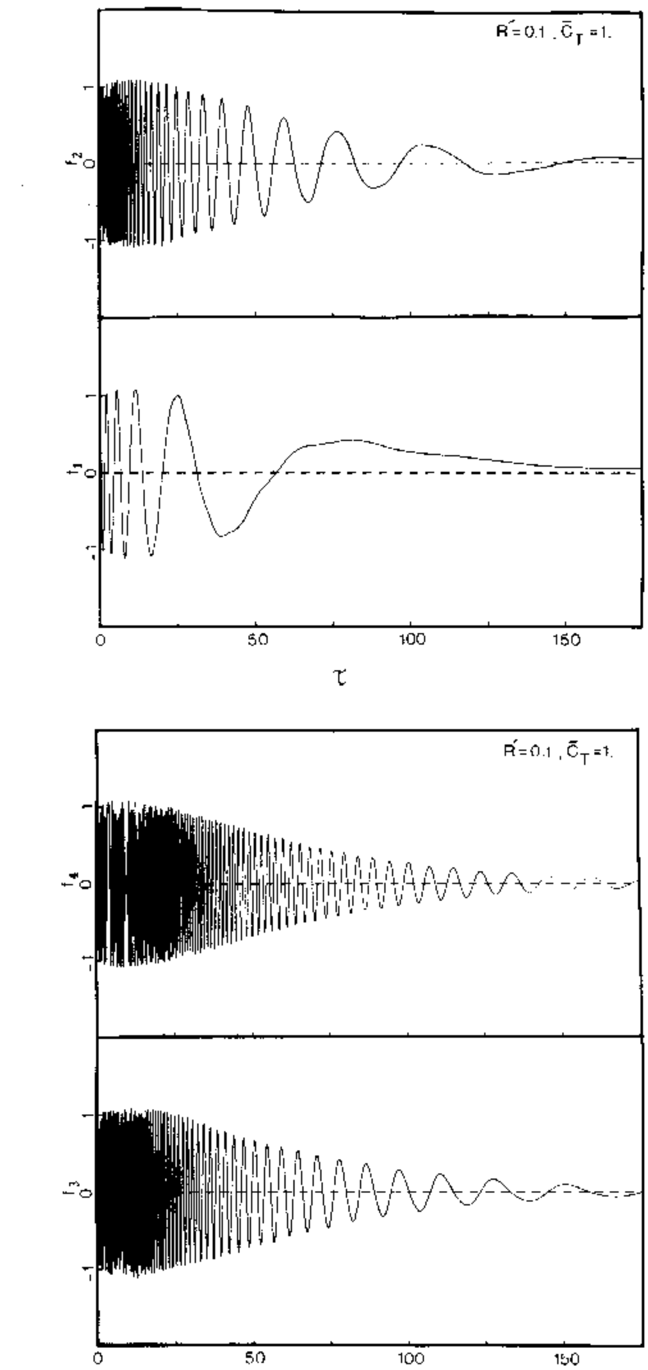

Fig. 3 Typical behavior of the beam during uniform extension, $R' = 0.1$: a) first two modes, b) third and fourth modes.

diameter, 1 mm thickness, and 50 cm initial length, $R' = 1$ corresponds to $\dot{l} = 2.5$ m/s.

It was observed that the free-end tension due to form drag, i.e. nonzero $\bar{C}_T$, has a destabilizing effect, which agrees with Ref. 10.

Exponential Extension

The dynamics of the beam was also investigated for the case of exponential extension, i.e.,

$$R' = V_0 \exp(-k\tau) \tag{15a}$$

where V_0 and k are constants. Clearly,

$$R = 1 + (V_0/k)\,[1 - \exp(-k\tau)] \tag{15b}$$

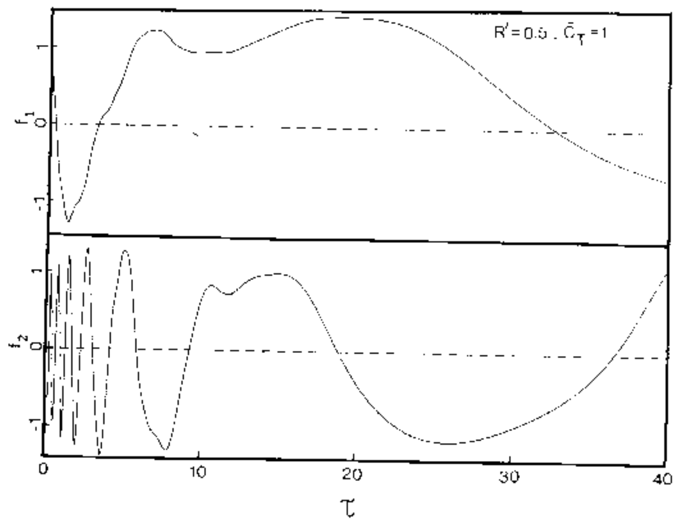

Fig. 4 Typical behavior of the beam during uniform extension, $R' = 0.5$.

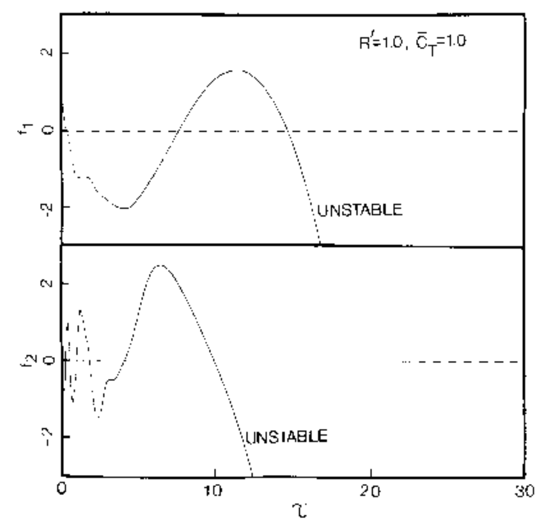

Fig. 5 Typical behavior of the beam during uniform extension, $R' = 1.0$.

Table 1 Dimensionless length of instability of the first mode during uniform extension

R'	0.5	0.8	1.0	1.5	2.0	2.5
R	stable	19.8	15.65	13.55	15.3 (!)	11.3

Table 2 Dimensionless length of instability of the first mode during several equivalent exponential extensions

k	-0.20	-0.10	-0.05	0	0.05	0.10
V_0	0.08695	0.33416	0.59910	1.0	1.5491	2.23416
R	16.50	18.88	16.02	15.65	12.60	13.07

The beam was extended to twenty times its original length within an interval $\tau = 0$ to $\tau = 19$ with various combinations of V_0 and k. All these combinations correspond to uniform extension of the beam to $R = 20$ within the same time interval with $R' = 1$. The effects of V_0 and k on the instability lengths for the extension under consideration are shown in Table 2. It may be noted that exponential extension is most stable when a small negative value of k (of the order of -0.10) is chosen. A positive value of k makes the exponential extension less stable than uniform extension (equivalent to $k = 0$).

Concluding Remarks

The main features of the analysis may be summarized as follows:

1) Numerical solutions for the dynamics of a beam moving axially in a fluid are obtained for uniform and exponential rates.

2) The beam can become unstable for much smaller values of extension rates compared to the critical velocity of a fixed-length beam in a uniform axial flow.

3) Exponential extension with small negative values of k is more stable than uniform extension.

4) Free-end tension due to form drag has a destabilizing effect on the extending beam.

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